12th International Congress on Mathematical Education
Program Name XX-YY-zz (pp. abcde-fghij)
8 July – 15 July, 2012, COEX, Seoul, Korea(This part is for LOC use only. Please do not change this part.)

ON THE GOLDEN RATIO

Michel Spira

Universidade Federal de Minas Gerais

michelspira@gmail.com

In this article we discuss some ideas associated with the Golden Ratio and its alleged appearances in art and biology.

Keywords: *golden ratio, golden number, geometry*

INTRODUCTION

The *Golden Ratio* is one of the most famous numbers. One gets used to "seeing" this number everywhere: in the Parthenon and in the Great Pyramids, in the proportions of the human body, in the Nautilus shell and so on. The aim of this article is to present a somewhat skeptical view on this.

GENERALITIES ABOUT Φ

We begin our work by having a superficial look at the golden ratio.

In Euclid's theory of areas, one finds Proposition 11 in Book II of ***The Elements***: *To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment*. In the notation of the picture below, one is asked, given *AB*, to find *P* such that the area of the rectangle on *AB* and *BC* (with $PB = BC$) is equal to the square on *AP*.

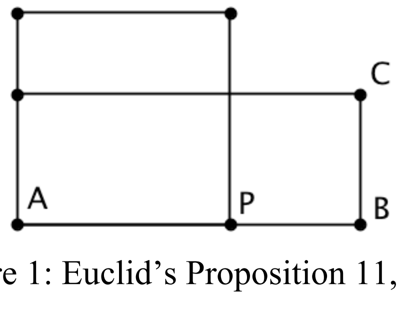


Figure 1: Euclid's Proposition 11, Book II

Setting $AP = x$ and $PB = y$, this is the same as

$$x^2 = y(x + y) \quad (1)$$

One is not trying to find lengths here (since one does not know the length of *AB*), but rather the position of *P*; in other words, one wants to find the ratio $\frac{x}{y}$ in which *P* divides *AB*. We do so by rewriting Eq.(1) as

$$\left(\frac{x}{y}\right)^2 = \frac{x}{y} + 1. \tag{2}$$

This ratio, which following current usage we will denote it by Φ, was explicitly defined and called *extreme and mean ratio* by Euclid in Book VI of the *Elements*. There it is presented as the solution to the problem of finding a point *P* which divides a given segment *AB* in a larger part *AP* and a smaller part *BP* such that *the whole is to the larger part as the larger part is to the smaller part*. If $AB = a$ and $AP = x$, one is asking for

$$\frac{a}{x} = \frac{x}{a-x} = \frac{1}{\frac{a}{x} - 1} \tag{3}$$

and one gets Eq.(2) with $\frac{a}{x}$ in place of $\frac{x}{y}$.

In passing, we note that the term *golden section* for Φappears to have been used for the first time by Martin Ohm in 1835, in his textbook ***Die Reine Elementar-Mathematik***; before him, Φ was called *divine proportion* by Luca Paccioli in his ***De divina proportione***, in 1509[1]. One will also find *golden mean* and *golden number* as terms for Φ.

We now rewrite Eq.(2) as

$$\Phi^2 = \Phi + 1 \tag{4}$$

or, equivalently, as

$$\frac{1}{\Phi} = \Phi - 1 \tag{5}$$

and we get the well-known properties of Φ. We remark that (3) and (4) follow from the very definition of Φ; in other words, they are not "extraordinary" or "mystical" properties of Φ, but only equivalent ways of saying thatΦ is the positive root of the polynomial

$$f = x^2 - x - 1. \tag{6}$$

The polynomial (5) is the minimal polynomial of Φ over $\mathbb{Q}$, and will be referred to as *f* in what follows. We see that Φ is an algebraic number (in fact, an algebraic integer) and, as is well known, all the algebraic properties of an algebraic number follow from its minimal polynomial; it is Φ's "luck" to have been defined by such a simple polynomial. To belabor

[1]Leonardo da Vinci illustrated this book, a fact that gave rise to the legend that da Vinci knew about Φ and used it in his works.

the point, Φ was defined as the unique positive number which satisfies (4); one should not be surprised it does so.

In this respect, it is quite sad to find (4) and (5) presented, in some texts written by people who should know better, as follows: first the author somehow gets to Eq.(6), finds

$$\Phi = \frac{1+\sqrt{5}}{2} = 1{,}618... \tag{7}$$

and then "computes"

$$(1{,}618...)^2 = 2{,}618... = 1 + 1{,}618... \tag{8}$$

with a similar computation for (5). An alternative approach is to define Φ by (7) and then proceed immediately to (8).
Let's go back toΦ. We first note that (4) gives rise, by repeated substitution, to the following interesting expression

$$\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\ldots}}} \tag{9}$$

and, similarly, (5) gives rise to

$$\Phi = 1 + \cfrac{1}{1+\cfrac{1}{1+\ldots}} \tag{10}$$

Φ is also related to another geomerical problem. Referring to the picture below

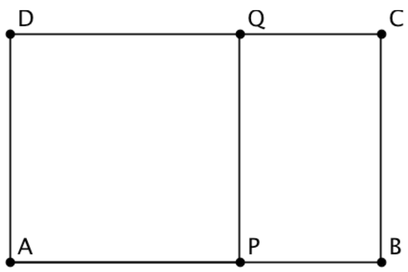


Figure 2: Removing a square from a rectangle

one asks for a rectangle *ABCD* such that if one cuts off a square *APQD,* the remaining rectangle *BCQP* is similar to the original one. It is easy to show that

$$\frac{AP}{BP} = \Phi \tag{11}$$

One can also ask how to construct Φ with ruler and compass; the easiest construction (as far as the author knows) is given below.

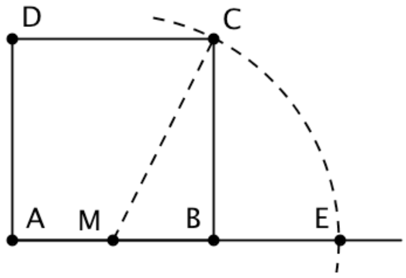


Figure 3: Constructing Φ

In this figure, *ABCD* is a square of side 1 and *M* is the midpoint of *AB*; one draws $\mathcal{C}(M, MC)$ and finds *E*. A quick computation, involving nothing more than Pythagoras's theorem, shows that $AE = \Phi$. More generally, if one starts with a square, this construction gives $\frac{AE}{AD} = \Phi$ and, by completing the rectangle on *A*, *E* and *D*, gives us a *golden rectangle*– a rectangle in which the ratio of the largest side to the smallest one is Φ.This utterly simple construction, relying only on a midpoint and an obvious circle, shows that every time an artist or an architect uses a square in his/her work, chances are that Φand/or a golden rectangle will make their unavoidable appearance somewhere, regardless of the fact that he/she knows something about Φ.

A common belief is that the golden rectangle is the most beautiful one. Presumably, this means that, given the choice between various rectangles of different proportions, people will favor the golden one or a close approximation. To check this, it is enough to make a survey and tabulate the results. This was done by Markowsky (1992); the most popular rectangle has seems to be the one with sides in 1.83 proportion, longest side in the horizontal position. In Markowsky's article one can also find templates to run one´s own experiment on which rectangle people prefer.

We point out that the initial step this construction makes $\sqrt{5}$ enter the picture; in fact, we have $MC = \frac{\sqrt{5}}{2}$ which is all one needs to get Φ; the circle just contributes with some simple algebra in order to get (7).

Another problem is how to divide a given segment in the golden ratio. To this end, there is an easy ruler and compass construction, which can be done easily by paper folding.[2]

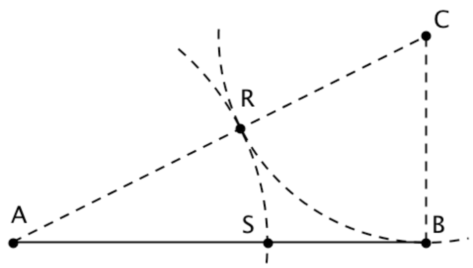


Figure 4: Dividing a segment in the golden ratio

[2]Provided the reader has no trouble using corners of square paper as folding points!

In this figure, BC is perpendicular to AB and $BC = \frac{1}{2}AB$. One first draws $\mathcal{C}(C, CB)$, determining R, and then $\mathcal{C}(A, AR)$, determining S. Again an easy computation shows that S divides AB in the golden ratio.

A well-known appearance of Φ is in the pentagon. In the figure below, we have a pentagon and one of its diagonals.

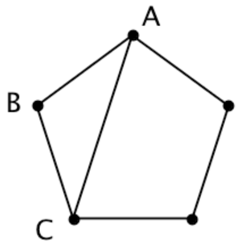


Figure 5: The pentagon and the golden ratio

One can show easily that

$$AC - AB = \frac{1}{AC}. \tag{12}$$

from which it follows that $AC = \Phi$.

Φ also makes a quite unexpected appearance in some other contexts. Among those, we choose first to show that there is Φ in an equilateral triangle. In the following figure, ABC is an equilateral triangle and M and N are midpoints of the corresponding sides; the reader can easily show that N divides MP in the golden ratio.

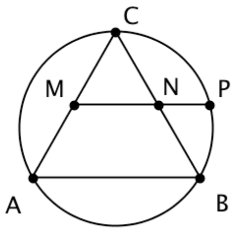


Figure 6: Φ in an equilateral triangle

Now we look at the picture below, where $ABCD$ is an arbitrary rectangle, and ask what are the conditions on R and S for the shaded triangles to have the same area.

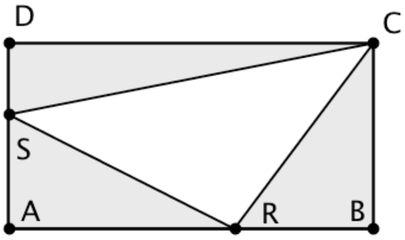


Figure 7: Φ in an arbitrary rectangle

You guessed it: this happens if and only if R and S divide AB and AD, respectively, in the golden ratio.

Also well-known, but still unexpected, is the relation between the Fibonacci sequence $1,1,2,3,5,8,13,21,\ldots$ and the golden ratio: the successive quotients $\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\ldots$ give better and better approximations for Φ. Having in view that divisions in 5 and 8 parts are quite common in art and architecture, it is no surprise that measurements of proportions in paintings, sculptures and edifications show, quite often, that (good approximations for) Φ and/or Φ^{-1} are hidden there.

To finish this section, we talk briefly about the famous *golden spiral*, which we show below.

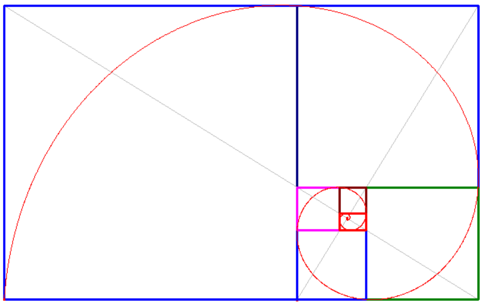


Figure 8: The golden spiral

It starts with a golden rectangle form which we extract a square and iterate this construction on each remaining rectangle. There is a unique (up to similarity and rigid motions) logarithmic spiral, which passes through the division points as above; this is the so-called golden spiral. Seeing a logarithmic spiral in nature, be it on nautilus shells, a sunflower or a galaxy is enough, for most people, to say that Φ is hiding there; this is the origin of the idea that Φ is an important ingredient in the inner works of Earth and Cosmos.

Φ AND SOME OF ITS ASSOCIATED MYTHS

The reader has seen that Φ is, indeed, a wonderful number, full of beautiful properties and capable of the most unexpected appearances. We now try to put this in a wider perspective.

First we go back to Figure 2; there we cut off a square from a rectangle and asked where *P* should be so that the remaining rectangle is similar to the original one. We now think of cutting off not a square, but a rectangle *APQD* such that $\frac{AP}{AD} = p$; Figure 2 corresponds to the particular case $p = 1$. It is straightforward to show that $\frac{AP}{AD}$ is the positive root of the polynomial

$$f_p := x^2 - px - 1. \tag{13}$$

We call this root Φ_p, so that Φ is now a member of the family $\{\Phi_p : p \geq 0\}$, corresponding to $p = 1$; we refer to a member of this family by *generalized golden number*. Φ_p can be constructed as in Figure 3, starting now with a rectangle *ABCD* with $AB = p$ and $AD = 1$.

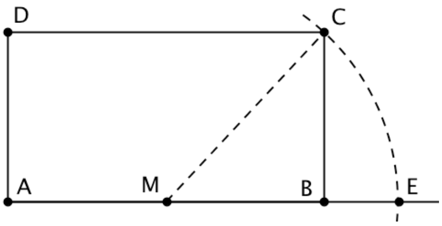


Figure 9: The construction of the generalized golden ratio

In this figure we have $AE = \Phi_p$. It is immediate to see, from the form of f_p, that the generalized golden numbers have the following properties:

$$\Phi_p^2 = p\Phi_p + 1 \tag{14}$$

$$\frac{1}{\Phi_p} = \Phi_p - p \tag{15}$$

$$\Phi_p = \sqrt{1 + p\sqrt{1 + p\sqrt{1 + \ldots}}} \tag{16}$$

and

$$\Phi_p = p + \cfrac{1}{p + \cfrac{1}{p + \ldots}} \tag{17}$$

Setting $p = 1$ in the above expressions we recover (4), (5), (9) and (10) for $\Phi = \Phi_1$. From this point of view, Φ`s properties are nothing special – infinitely many numbers have quite similar properties.

One might argue that Φ has, say, a wonderful relation to the pentagon, given by (12), which is not shared by any of the Φ_p. But consider a regular *n*-gon with odd $n \geq 5$ and side 1, as in the figure below.

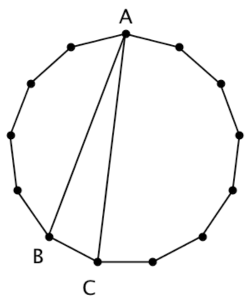


Figure 10: A polygon with an odd number of sides and Φ_p

Letting *AC* be the longest segment connecting two vertices and *AB* the second longest such, it is a neat exercise to show that we recover (12) in exactly the same form. Hence, if

$AB = p$ then $AC = \Phi_p$; the particular case $n = 5$ (when $AB = 1$) gives us Φ. We now see that Φ`s relation to the pentagon is just a special case of a much more general phenomenon.

But certainly Φ`s relation with the logarithmic spiral is a special one! Not quite. Logarithmic spirals come in a family indexed by p and can be constructed from any $\Phi_p \times 1$ rectangle, in exactly the same way we constructed the golden spiral in Figure 8; one removes rectangles similar to $p \times 1$ ones. Below we present some examples; the numbers below the spirals are the corresponding values of Φ_p. The top right spiral is the golden one.

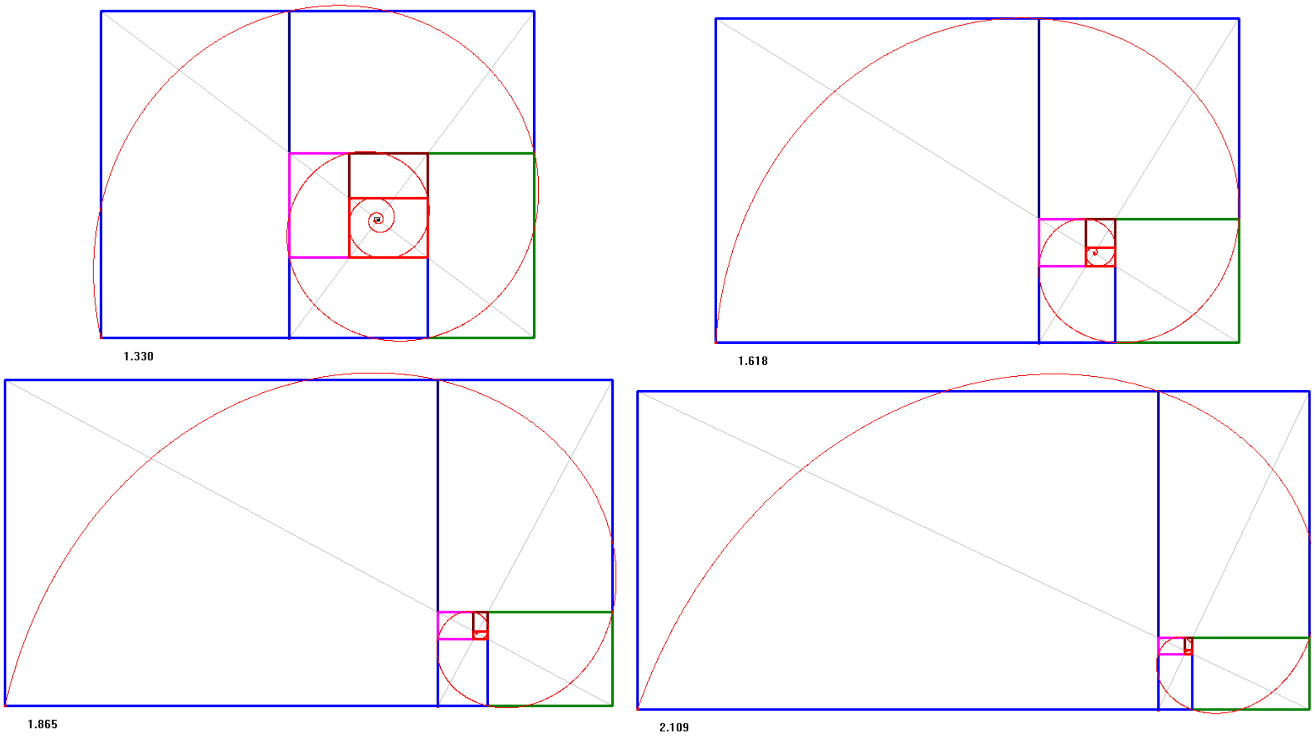


Figure 11: Some logarithmic spirals

Notice that logarithmic spirals are not, in general, tangent to the sides of their generating rectangle. In fact, there is only one such; we leave it to the reader to find out the corresponding Φ_p.

It is simply not true that all spirals in nature are golden ones. Even the prime such “example” is false; the nautilus shell corresponds to a spiral with $p = 1{,}33$ (the first one in Figure 11).

Figure 12: Golden spiral (left) and nautilus shell (right)

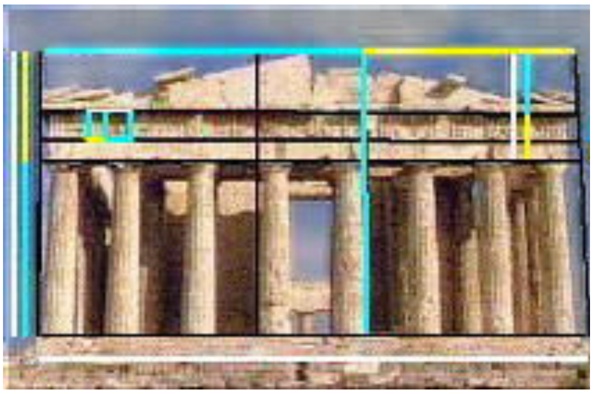
Finally, what about the relation between Φ and the Fibonacci numbers? Well, there is no such. The fact is that $a,b>0$ are arbitrary and the sequence (a_n) is given by $a_1=a$, $a_2=b$ and $a_n=a_{n-1}+a_{n-2}$ for $n\geq 2$, it is true that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\Phi$. This happens just because of the special form of the recursion, whose associated polynomial is x^2-x-1. So Fibonacci has nothing to say here.

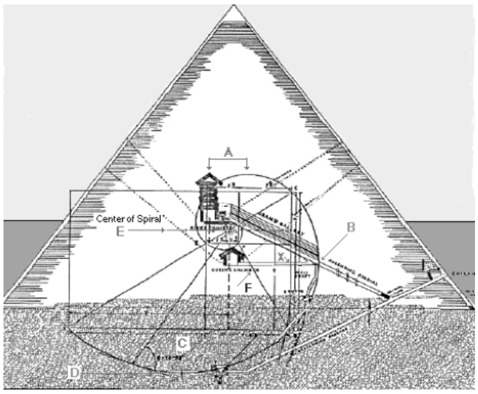
We could go on forever about the myths surrounding the golden ratio, but time and space force us to stop here. Hopefully, the message is clear: most, if not all, of what is said about Φ and its unique properties is false, and can be refuted by elementary mathematics and a dose of common sense.

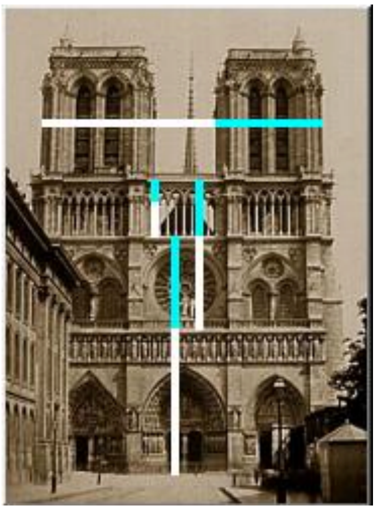
HOW TO FIND Φ

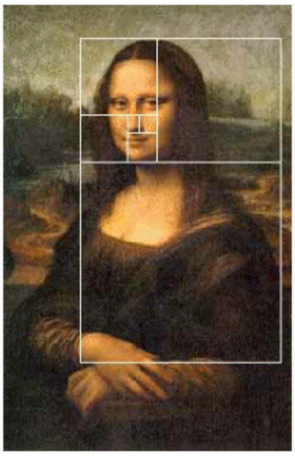
In this section we will, by presenting a few pictures chosen among uncountable similar ones, show that the quest for Φ can go a long way. In all these pictures, the claim is that Φ is there. No details will be given; we just call the reader's attention to the following, often in combination:

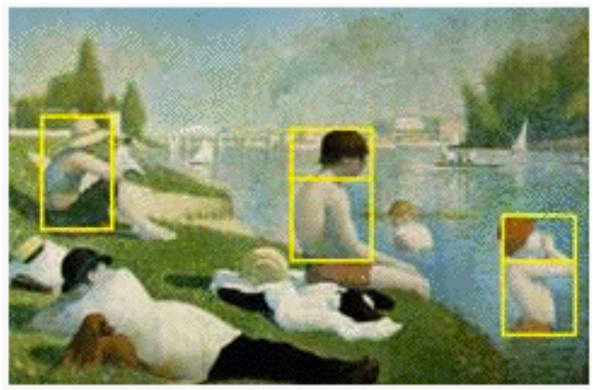
- arbitrary placement of points, lines, rectangles and spirals;
- arbitrary thickness of points and lines used as basis for measurements (allows for easy fudging, so that $\frac{5}{3}$, say, can be taken as Φ);
- measurements of monuments eroded by time and of objects in photographs distorted by perspective.

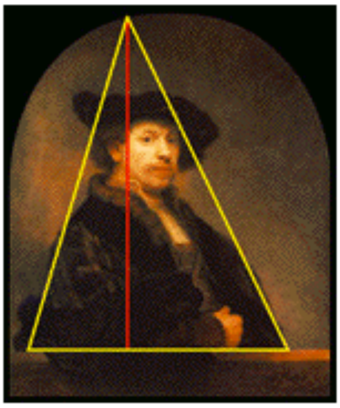


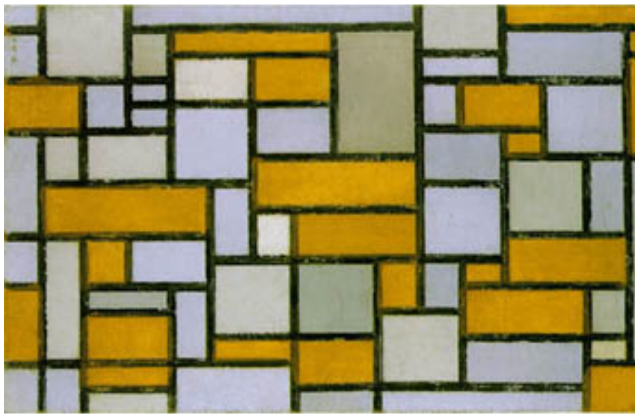

A
Center of Spiral
E
B
F
C
D

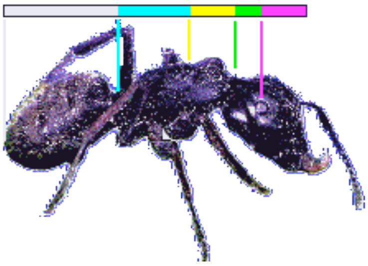


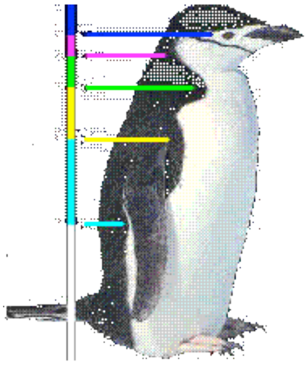


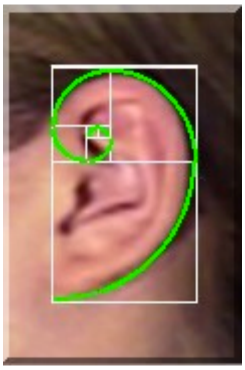


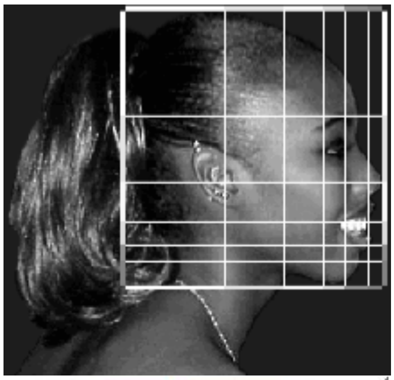


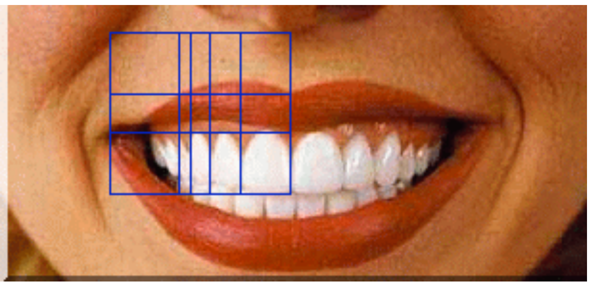


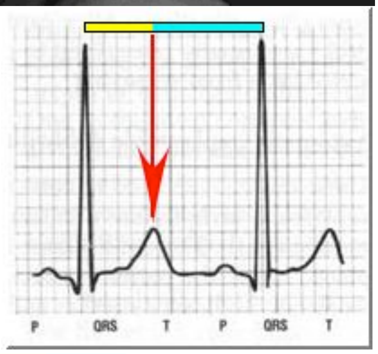


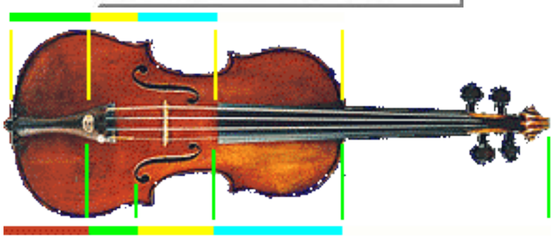









P
QRS
T
P
QRS
T



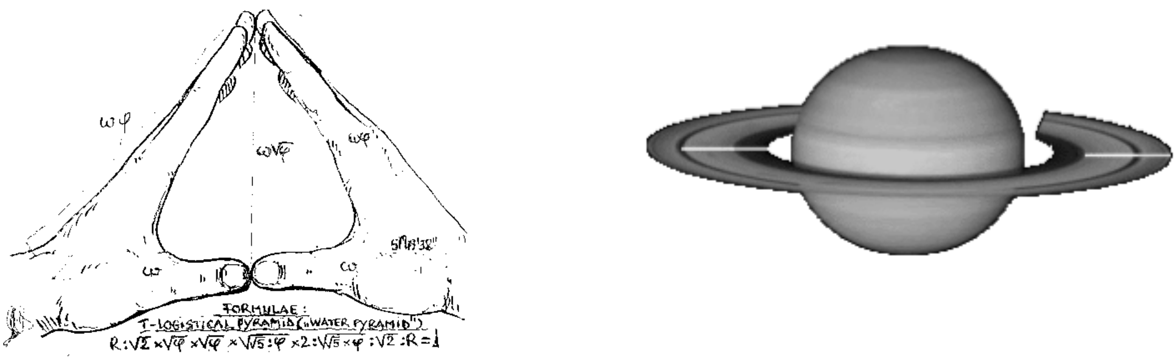


Figure 13: Φ is everywhere

One could go on forever, but it is time to move on to the conclusion of this article.

Φ, BEAUTY AND SPIRITUAL DEVELOPMENT

When doing basic Google search for the golden ratio and related topics, the following exercise sheet struck the author.

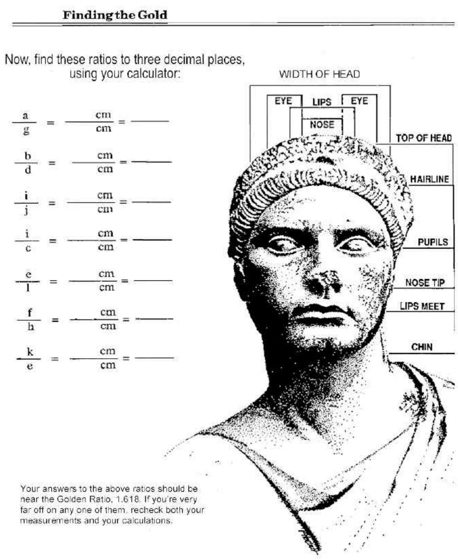


Figure 14: Φ is there, you better believe it

The title is "finding the gold". In it, kids are asked to make some measurements with a ruler between previously marked points and to compute a few quotients. The text on the lower left is worth transcribing: "Your answers to the above ratios should be near the Golden Ratio 1.618. If you're very far off on any one of them, recheck both your measurements and your calculations." And this to three decimal places!

This is the voice of authority speaking; it's there, you better find it, no questioning allowed. This may sound harmless enough, but it is not. If you are a mathematician, Math teacher or scientist and say something that involves numbers, people will believe you; this power can and has been misused.

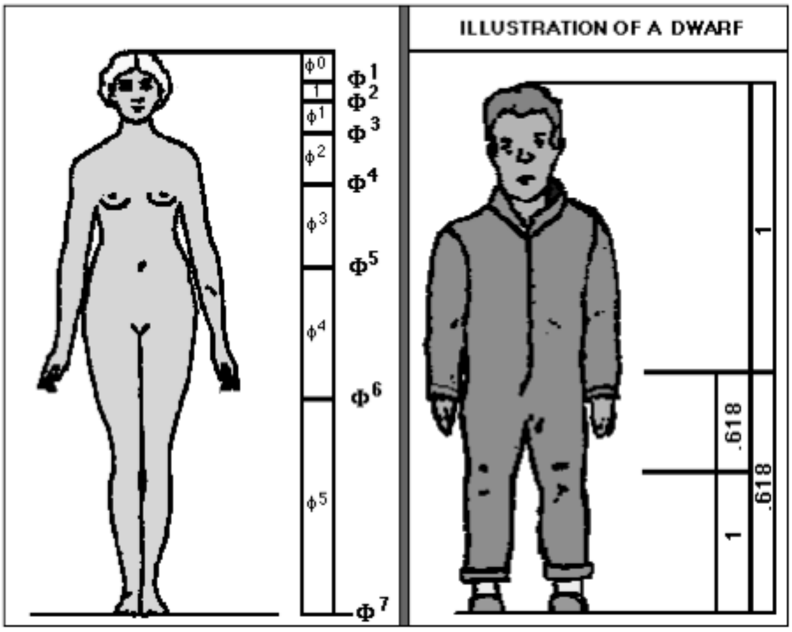
On the harmless side, one finds the following statements, written in a book about the golden ratio and translated to the best of the author's ability:

Anything that breaks in half can be repaired, but if it reaches the $\frac{1}{\Phi}$ mark no repair will be possible (do you believe this?).

A fruit tree will have the most succulent fruit when it reaches $\frac{1}{\Phi}$ of its total load.

A woman's cycle lasts 28 days, therefore $\frac{1}{\Phi}$ of 28 will be 17,5 days, when fertilization is guaranteed.

Still harmless is the famous connection between Φ and beauty. As the theory goes, you should measure your navel ratio, i.e., the ratio in which your navel divides your height (from feet to head, in this order). The closer this ratio is to Φ, the more beautiful you are. Of course a navel, having a diameter, is not a point – a lot of fudging can be done here.



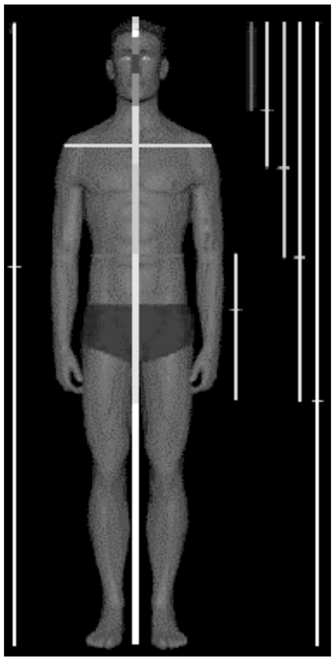


Figure 15: Φ and beauty (look at her navel!)

Well, this is silly enough. But what if "beauty" is replaced by "spiritual development"? Still sounds silly, but then one finds Neroman (1989, originally 1940), written by someone who speaks with "authority"; even high school algebra and geometry can intimidate people. The idea is as follows. Human kind is in permanent state of spiritual development, the degree of which in a given race can be measured by the navel ratio of this race's women. This ratio is always less than Φ, since (of course!) Φ represents perfection.

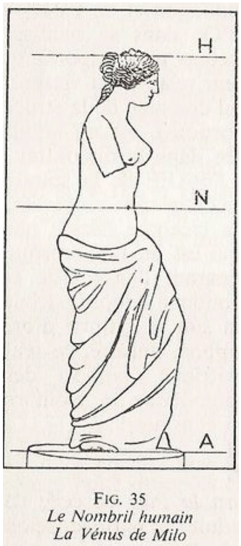


Figure 16: Perfect "spiritual development"

A few freehand drawings of women of various races (rigorously true to life, we are told) follow, a horizontal line showing where the subject is divided in the golden ratio so as to provide a visual estimation of how far the navel is from the ideal position – in other words, how far from the ideal spiritual development the corresponding race is. These drawings are shown in the next figure.

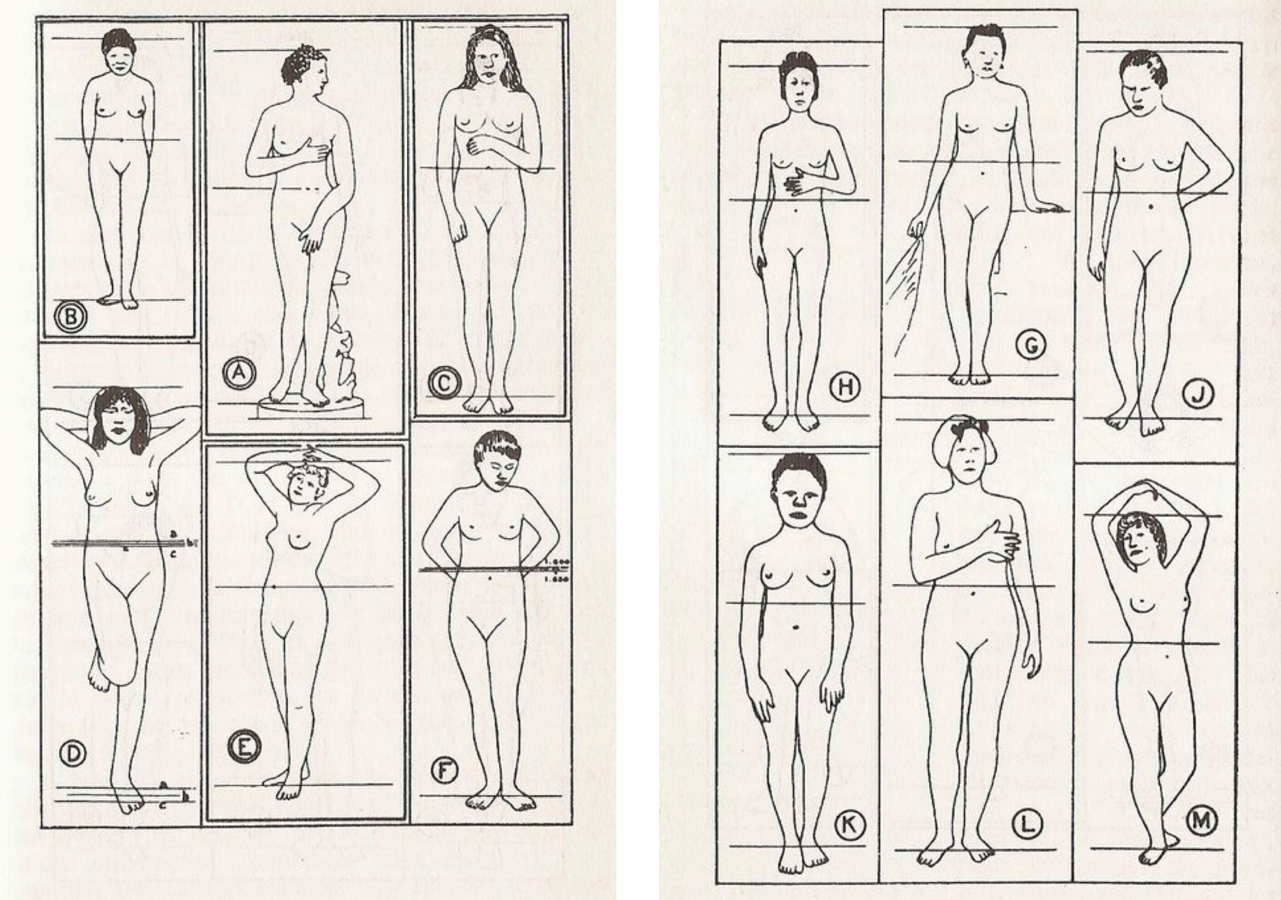


FIG. 35 *bis*. — *Le Nombril humain*
A) La Vénus de Médicis. — *B)* Femme Hottentote métissée (tribu Korana, Afrique Australe). — *C)* Femme Tamoule (Ceylan). — *D)* Jeune femme arabe de Tunis. — *E)* Femme Wende (slave). — *F)* Cinghalaise de Ceylan.

FIG. 35 *ter*. — *Le Nombril humain*
G) Jeune siamoise. — *H)* Femme Amahara (Abyssinie). — *J)* Femme malaise de Batavia. — *K)* Jeune Négrille (Afrique équatoriale). — *L)* Hollandaise de Scheveningue. — *M)* Femme juive.

Figure 17: Comparison of the "spiritual development" of various races

Can the reader guess which races are at the bottom of the spiritual development scale? Of course! Jews and Blacks, no doubt about it. Neromam takes pains to tell us so, in case we did not notice: "… cet écart sur la divine poportion est surtout accusé chez la Juive (fig. M) et chez la jeune Négrille de l'Afrique équatoriale (fig. K)".

Now this is certainly not harmless. One is reminded of the uses of I.Q. tests to typify people as morons; those interested should read Gould (1981). The idea is the same; one ranks people by a number (the score in I.Q. tests or the navel ratio) to which is attributed a meaning, which almost invariably will be used to vindicate existing prejudice and social divisions.

Gould points out that the I.Q. rank had as one of its consequences the forced sterilization of some of those characterized as morons, as well as the establishment of immigration quotas in the USA; these quotas caused the death of thousands in concentration camps during World War II. Paraphrasing Gould, one can say that sometimes Math is more powerful than swords.

It is our responsibility, as mathematicians and teachers, to fight the use of Math as a language of power. A good place to start is to tell people that Φ, golden numbers cultists nowithstanding, is not "the key to the living world" and that most of what is said about it is bogus. The author hopes that this article will help people to do so.

References

Devlin, K. Good stories, pity they're not true. http://www.maa.org/devlin/devlin_06_04.html

Falbo, C. The golden ratio: a contrary viewpoint. http://www.sonoma.edu/math/faculty/falbo/cmj123-134

Falbo, C. Generalizations of the golden ratio http://www.mathfile.net/generalized_phi_mathpage.pdf

Gould, S. J. (1981). The Mismeasure of Man. W. W. Norton & Company

Herz-Fischler, R. (1987). A Mathematical History of the Golden Number. Dover Publications, Inc.

Huntley, H. E. (1970). The divine proportion: a study in mathematical beauty. Dover Publications, Inc.

Markowsky, G., O. (1992). Misconceptions about the golden ratio. *The College Mathematics Journal*, Vol 23, no.1, pp. 2-19